6.4 The Logistic Equation

- Solve and analyze logistic differential equations.
- Use logistic differential equations to model and solve applied problems.

Logistic Differential Equation

In Section 6.2, the exponential growth model was derived from the fact that the rate of change of a variable y is proportional to the value of y. You observed that the differential equation $dy/dt = ky$ has the general solution $y = Ce^{kt}$. Exponential growth is unlimited, but when describing a population, there often exists some upper limit L past which growth cannot occur. This upper limit L is called the **carrying capacity,** which is the maximum population $y(t)$ that can be sustained or supported as time t increases. A model that is often used to describe this type of growth is the **logistic differential equation**

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$ Logistic differential equation

where k and L are positive constants. A population that satisfies this equation does not grow without bound, but approaches the carrying capacity L as t increases.

From the equation, you can see that if y is between 0 and the carrying capacity L, then $dy/dt > 0$, and the population increases. If y is greater than L, then $dy/dt < 0$, and the population decreases. The general solution of the logistic differential equation is derived in the next example.

EXAMPLE 1 Deriving the General Solution

Solve the logistic differential equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right)$.

Solution Begin by separating variables.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$ Write differential equation.

$$\frac{1}{y(1 - y/L)}\,dy = k\,dt$$ Separate variables.

$$\int \frac{1}{y(1 - y/L)}\,dy = \int k\,dt$$ Integrate each side.

$$\int \left(\frac{1}{y} + \frac{1}{L - y}\right)dy = \int k\,dt$$ Rewrite left side using partial fractions.

$$\ln|y| - \ln|L - y| = kt + C$$ Find antiderivative of each side.

$$\ln\left|\frac{L - y}{y}\right| = -kt - C$$ Multiply each side by -1 and simplify.

$$\left|\frac{L - y}{y}\right| = e^{-kt - C} = e^{-C}e^{-kt}$$ Exponentiate each side.

$$\frac{L - y}{y} = be^{-kt}$$ Let $\pm e^{-C} = b$.

Solving this equation for y produces $y = \dfrac{L}{1 + be^{-kt}}$. ■

EXPLORATION

Use a graphing utility to investigate the effects of the values of L, b, and k on the graph of

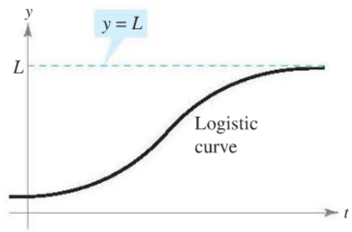
$$y = \frac{L}{1 + be^{-kt}}.$$

Include some examples to support your results.

From Example 1, you can conclude that all solutions of the logistic differential equation are of the general form

$$y = \frac{L}{1 + be^{-kt}}.$$

The graph of the function y is called the *logistic curve*, as shown in Figure 6.21. In the next example, you will verify a particular solution of a logistic differential equation and find the initial condition.



Note that as $t \to \infty$, $y \to L$.
Figure 6.21

EXAMPLE 2 Verifying a Particular Solution

Verify that the equation

$$y = \frac{4}{1 + 2e^{-3t}}$$

satisfies the logistic differential equation, and find the initial condition.

Solution Comparing the given equation with the general form derived in Example 1, you know that $L = 4$, $b = 2$, and $k = 3$. You can verify that y satisfies the logistic differential equation as follows.

$$y = 4(1 + 2e^{-3t})^{-1} \qquad \text{Rewrite using negative exponent.}$$

$$y' = 4(-1)(1 + 2e^{-3t})^{-2}(-6e^{-3t}) \qquad \text{Apply Power Rule.}$$

$$= 3\left(\frac{4}{1 + 2e^{-3t}}\right)\left(\frac{2e^{-3t}}{1 + 2e^{-3t}}\right) \qquad \text{Rewrite.}$$

$$= 3y\left(\frac{2e^{-3t}}{1 + 2e^{-3t}}\right) \qquad \text{Rewrite using } y = \frac{4}{1 + 2e^{-3t}}.$$

$$= 3y\left(1 - \frac{1}{1 + 2e^{-3t}}\right) \qquad \text{Rewrite fraction using long division.}$$

$$= 3y\left(1 - \frac{4}{4(1 + 2e^{-3t})}\right) \qquad \text{Multiply fraction by } \frac{4}{4}.$$

$$= 3y\left(1 - \frac{y}{4}\right) \qquad \text{Rewrite using } y = \frac{4}{1 + 2e^{-3t}}.$$

So, y satisfies the logistic differential equation $y' = 3y\left(1 - \frac{y}{4}\right)$. The initial condition can be found by letting $t = 0$ in the given equation.

$$y = \frac{4}{1 + 2e^{-3(0)}} = \frac{4}{3} \qquad \text{Let } t = 0 \text{ and simplify.}$$

So, the initial condition is $y(0) = \frac{4}{3}$. ■

EXAMPLE 3 Verifying the Upper Limit

Verify that the upper limit of $y = \dfrac{4}{1 + 2e^{-3t}}$ is 4.

Solution In Figure 6.22, you can see that the values of y appear to approach 4 as t increases without bound. You can come to this conclusion numerically, as shown in the table.

t	0	1	2	5	10	100
y	1.3333	3.6378	3.9803	4.0000	4.0000	4.0000

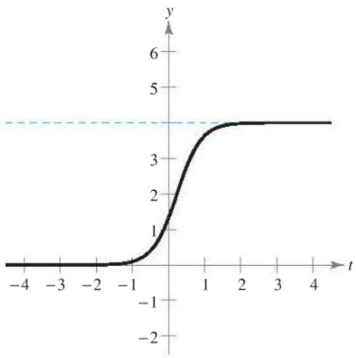


Figure 6.22

Finally, you can obtain the same results analytically, as follows.

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{4}{1 + 2e^{-3t}} = \frac{\lim\limits_{t\to\infty} 4}{\lim\limits_{t\to\infty}(1 + 2e^{-3t})} = \frac{4}{1 + 0} = 4$$

The upper limit of y is 4, which is also the carrying capacity $L = 4$.

EXAMPLE 4 Determining the Point of Inflection

Sketch a graph of $y = \dfrac{4}{1 + 2^{-3t}}$. Calculate y'' in terms of y and y'. Then determine the point of inflection.

Solution From Example 2, you know that

$$y' = 3y\left(1 - \frac{y}{4}\right).$$

Now calculate y'' in terms of y and y'.

$$y'' = 3y\left(-\frac{y'}{4}\right) + \left(1 - \frac{y}{4}\right)3y' \qquad \text{Differentiate using Product Rule.}$$

$$y'' = 3y'\left(1 - \frac{y}{2}\right) \qquad \text{Factor and simplify.}$$

When $2 < y < 4$, $y'' < 0$ and the graph of y is concave downward. When $0 < y < 2$, $y'' > 0$ and the graph of y is concave upward. So, a point of inflection must occur at $y = 2$. The corresponding t-value is

$$2 = \frac{4}{1 + 2e^{-3t}} \Rightarrow 1 + 2e^{-3t} = 2 \Rightarrow e^{-3t} = \frac{1}{2} \Rightarrow t = \frac{1}{3}\ln 2.$$

The point of inflection is $\left(\frac{1}{3}\ln 2, 2\right)$, as shown in Figure 6.23. ■

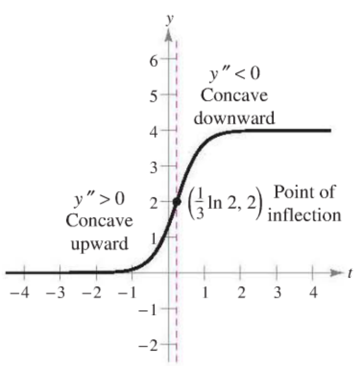


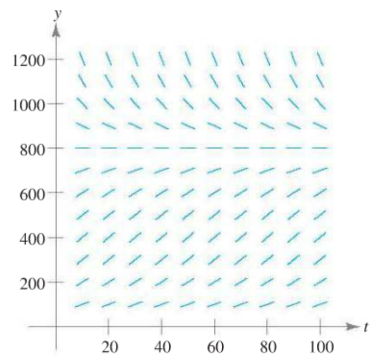
Figure 6.23

NOTE In Example 4, the point of inflection occurs at $y = \frac{L}{2}$. This is true for any logistic growth curve for which the solution starts below the carrying capacity L (see Exercise 37). ■

EXAMPLE 5 Graphing a Slope Field and Solution Curves

Graph a slope field for the logistic differential equation $y' = 0.05y\left(1 - \frac{y}{800}\right)$. Then graph solution curves for the initial conditions $y(0) = 200$, $y(0) = 1200$, and $y(0) = 800$.

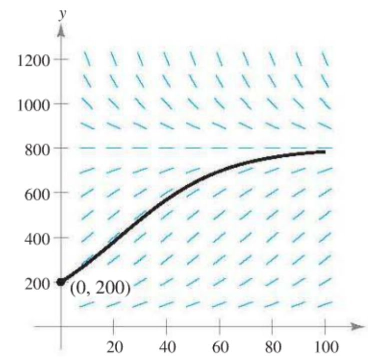
Solution You can use a graphing utility to graph the slope field shown in Figure 6.24. The solution curves for the initial conditions $y(0) = 200$, $y(0) = 1200$, and $y(0) = 800$ are shown in Figures 6.25–6.27.



Slope field for

$y' = 0.05y\left(1 - \frac{y}{800}\right)$

Figure 6.24

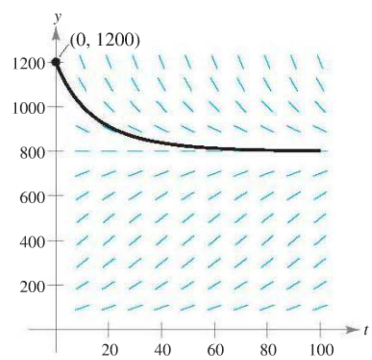


Particular solution for

$y' = 0.05y\left(1 - \frac{y}{800}\right)$

and initial condition $y(0) = 200$

Figure 6.25

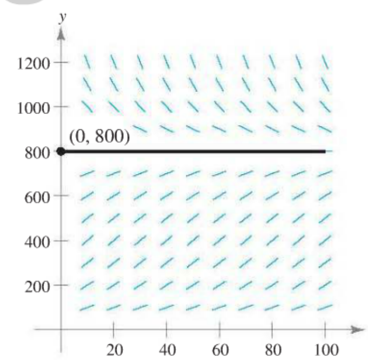


Particular solution for

$y' = 0.05y\left(1 - \frac{y}{800}\right)$

and initial condition $y(0) = 1200$

Figure 6.26



Particular solution for

$y' = 0.05y\left(1 - \frac{y}{800}\right)$

and initial condition $y(0) = 800$

Figure 6.27

Note that as t increases without bound, the solution curves in Figures 6.25–6.27 all tend to the same limit, which is the carrying capacity of 800. ■

EXAMPLE 6 Solving a Logistic Differential Equation

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right), \quad 40 \le p \le 4000$$

where t is the number of years.

a. Write a model for the elk population in terms of t.

b. Graph the slope field for the differential equation and the solution that passes through the point $(0, 40)$.

c. Use the model to estimate the elk population after 15 years.

d. Find the limit of the model as $t \to \infty$.

EXPLORATION

Explain what happens if $p(0) = L$.

Solution

a. You know that $L = 4000$. So, the solution of the equation is of the form

$$p = \frac{4000}{1 + be^{-kt}}.$$

Because $p(0) = 40$, you can solve for b as follows.

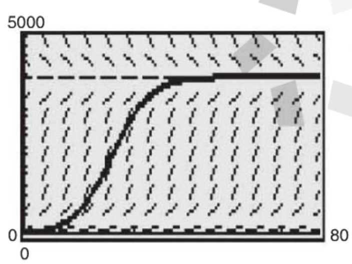
$$40 = \frac{4000}{1 + be^{-k(0)}}$$

$$40 = \frac{4000}{1 + b} \quad \Rightarrow \quad b = 99$$

Then, because $p = 104$ when $t = 5$, you can solve for k.

$$104 = \frac{4000}{1 + 99e^{-k(5)}} \quad \Rightarrow \quad k \approx 0.194$$

So, a model for the elk population is given by $p = \dfrac{4000}{1 + 99e^{-0.194t}}$.



Slope field for $\dfrac{dp}{dt} = 0.194p\left(1 - \dfrac{p}{4000}\right)$ and the solution passing through $(0, 40)$

Figure 6.28

b. Using a graphing utility, you can graph the slope field of

$$\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$$

and the solution that passes through $(0, 40)$, as shown in Figure 6.28.

c. To estimate the elk population after 15 years, substitute 15 for t in the model.

$$p = \frac{4000}{1 + 99e^{-0.194(15)}} \qquad \text{Substitute 15 for } t.$$

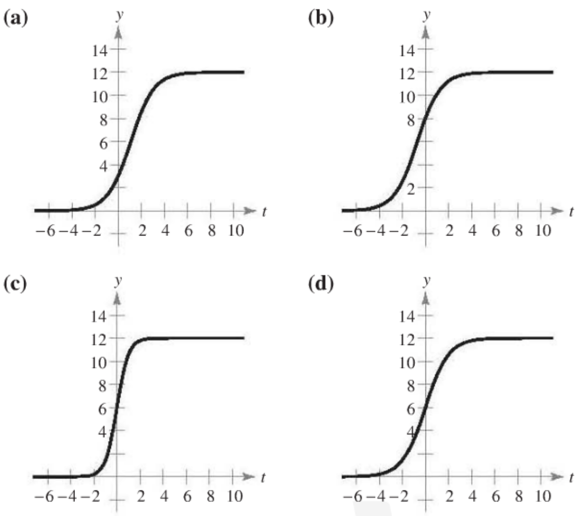
$$= \frac{4000}{1 + 99e^{-2.91}} \approx 626 \qquad \text{Simplify.}$$

d. As t increases without bound, the denominator of $\dfrac{4000}{1 + 99e^{-0.194t}}$ gets closer and closer to 1. So, $\displaystyle\lim_{t\to\infty} \frac{4000}{1 + 99e^{-0.194t}} = 4000$. ■

6.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, match the logistic equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



1. $y = \dfrac{12}{1 + e^{-t}}$

2. $y = \dfrac{12}{1 + 3e^{-t}}$

3. $y = \dfrac{12}{1 + \frac{1}{2}e^{-t}}$

4. $y = \dfrac{12}{1 + e^{-2t}}$

In Exercises 5–8, verify that the equation satisfies the logistic differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right).$$

Then find the initial condition.

5. $y = \dfrac{8}{1 + e^{-2t}}$

6. $y = \dfrac{10}{1 + 3e^{-4t}}$

7. $y = \dfrac{12}{1 + 6e^{-t}}$

8. $y = \dfrac{14}{1 + 5e^{-3t}}$

In Exercises 9–12, the logistic equation models the growth of a population. Use the equation to (a) find the value of *k*, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution *P*(*t*).

9. $P(t) = \dfrac{2100}{1 + 29e^{-0.75t}}$

10. $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

11. $P(t) = \dfrac{6000}{1 + 4999e^{-0.8t}}$

12. $P(t) = \dfrac{1000}{1 + 8e^{-0.2t}}$

CAS In Exercises 13–16, the logistic differential equation models the growth rate of a population. Use the equation to (a) find the value of *k*, (b) find the carrying capacity, (c) use a computer algebra system to graph a slope field, and (d) determine the value of *P* at which the population growth rate is the greatest.

13. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$

14. $\dfrac{dP}{dt} = 0.5P\left(1 - \dfrac{P}{250}\right)$

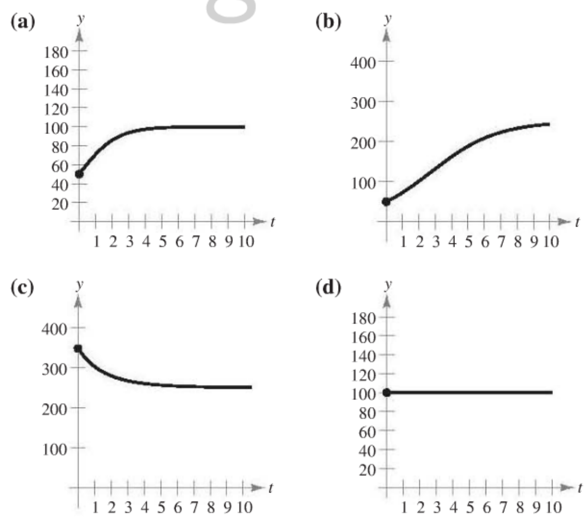
15. $\dfrac{dP}{dt} = 0.1P - 0.0004P^2$

16. $\dfrac{dP}{dt} = 0.4P - 0.00025P^2$

In Exercises 17–20, find the logistic equation that satisfies the initial condition. Then use the logistic equation to find *y* when *t* = 5 and *t* = 100.

	Logistic Differential Equation	*Initial Condition*
17.	$\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{36}\right)$	(0, 4)
18.	$\dfrac{dy}{dt} = 2.8y\left(1 - \dfrac{y}{10}\right)$	(0, 7)
19.	$\dfrac{dy}{dt} = \dfrac{4y}{5} - \dfrac{y^2}{150}$	(0, 8)
20.	$\dfrac{dy}{dt} = \dfrac{3y}{20} - \dfrac{y^2}{1600}$	(0, 15)

In Exercises 21–24, match the logistic differential equation and initial condition with the graph of its solution. [The graphs are labeled (a), (b), (c), and (d).]



21. $\dfrac{dy}{dt} = 0.5y\left(1 - \dfrac{y}{250}\right)$, (0, 350)

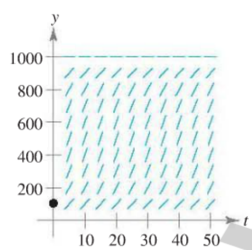
22. $\dfrac{dy}{dt} = 0.9y\left(1 - \dfrac{y}{100}\right)$, (0, 100)

23. $\frac{dy}{dt} = 0.5y\left(1 - \frac{y}{250}\right)$, $(0, 50)$

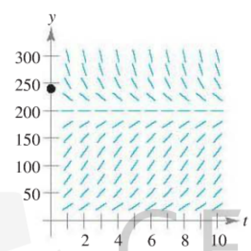
24. $\frac{dy}{dt} = 0.9y\left(1 - \frac{y}{100}\right)$, $(0, 50)$

Slope Fields **In Exercises 25–28, a logistic differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

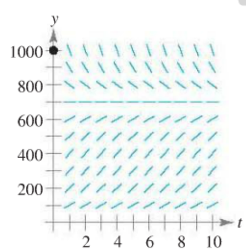
25. $\frac{dy}{dt} = 0.2y\left(1 - \frac{y}{1000}\right)$, $(0, 105)$



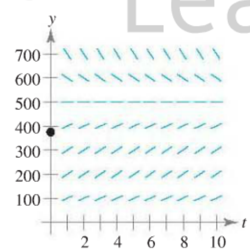
26. $\frac{dy}{dt} = 0.9y\left(1 - \frac{y}{200}\right)$, $(0, 240)$



27. $\frac{dy}{dt} = 0.6y\left(1 - \frac{y}{700}\right)$, $(0, 1000)$



28. $\frac{dy}{dt} = 0.4y\left(1 - \frac{y}{500}\right)$, $(0, 375)$



WRITING ABOUT CONCEPTS

29. Describe what the value of L represents in the logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$.

30. It is known that $y = \frac{L}{1 + be^{-kt}}$ is a solution of the logistic differential equation $\frac{dy}{dt} = 0.75y\left(1 - \frac{y}{2500}\right)$. Is it possible to determine L, k, and b from the information given? If so, find their values. If not, which value(s) cannot be determined and what information do you need to determine the value(s)?

31. Is the logistic differential equation separable? Explain.

CAPSTONE

32. Suppose the growth of a population is modeled by a logistic equation. As the population increases, its rate of growth decreases. What do you think causes this to occur in real-life situations such as animal or human populations?

33. ***Endangered Species*** A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

(a) Write a logistic equation that models the population of panthers in the preserve.

(b) Find the population after 5 years.

(c) When will the population reach 100?

(d) Write a logistic differential equation that models the growth rate of the panther population. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.

(e) After how many years is the panther population growing most rapidly? Explain.

34. ***Bacteria Growth*** At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 4 grams. The maximum weight of the culture is 20 grams.

(a) Write a logistic equation that models the weight of the bacterial culture.

(b) Find the culture's weight after 5 hours.

(c) When will the culture's weight reach 18 grams?

(d) Write a logistic differential equation that models the growth rate of the culture's weight. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.

(e) After how many hours is the culture's weight increasing most rapidly? Explain.

True or False? **In Exercises 35 and 36, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

35. For the logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$, if $y > L$, then $dy/dt > 0$ and the population increases.

36. For the logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$, if $0 < y < L$, then $dy/dt > 0$ and the population increases.

37. For any logistic growth curve, show that the point of inflection occurs at $y = \frac{L}{2}$ when the solution starts below the carrying capacity L.

38. Show that if $y = \frac{1}{1 + be^{-kt}}$, then $\frac{dy}{dt} = ky(1 - y)$.